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Multi-Dimensional Diffusion MRI Sampling Scheme: B-tensor Design and Accurate Signal Reconstruction

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Synopsis

b-tensor encoding enables the separation of isotropic and anisotropic tensors. However, little consideration has been given as to how to design a b-tensor encoding scheme. In this work, we propose the first 4D basis for representing the diffusion signal acquired with b-tensor encoding. We study the properties of the diffusion signal in this basis to give recommendations for optimally sampling the space of axisymmetric b-tensors. We show, using simulations, that the proposed sampling scheme enables accurate reconstruction of the diffusion signal by expansion in this basis using a clinically feasible number of samples.

Introduction

Multi-dimensional diffusion MRI (MD-dMRI) is a recent diffusion signal modelling and reconstruction framework where the diffusion signal is derived from the diffusion tensor distribution (DTD). Advanced magnetic gradient modulation schemes, resulting in b-tensor encoding, enable separation of isotropic and anisotropic tensors. Often diffusion tensors are assumed to be axisymmetric, so the DTD is 4D. Similarly, we use the spherical Laguerre basis, a 3D orthogonal basis with an exponential weighting function in the radial direction, for modulated with a Laguerre polynomial, which together form an orthogonal basis over the diffusion tensor distribution (DTD). Advanced magnetic gradient modulation schemes, resulting in b-tensor encoding, enable separation of isotropic b-tensors. We show, using simulations, that the proposed sampling scheme enables accurate reconstruction of the diffusion signal by expansion in this basis using a clinically feasible number of samples.

MD-dMRI Signal Basis and b-tensor Sampling Scheme

\[
\frac{S}{S_0}(\mathbf{B}) = \int P(D) \exp(-\mathbf{B} : D) dD = \langle \exp(-\mathbf{B} : D) \rangle,
\]

is the diffusion signal acquired with b-tensor encoding, where \(P(D)\) is the DTD, \(D, \mathbf{B}\) are second order symmetric positive-definite diffusion and b-tensors respectively, and \(\mathbf{B} : D = \sum_i \sum_j b_j D_{ij}\). For a discrete set of diffusion tensor populations,

\[
\frac{S}{S_0}(\mathbf{B}) = \sum_{k=1}^{N_D} w_k \exp(-\mathbf{B} : \mathbf{D}_k),
\]

where \(w_k\) is the proportion of tensor population \(\mathbf{D}_k\) and \(N_D\) is the number of populations. The axisymmetric diffusion tensor can be parameterised as,

\[
\mathbf{D} = R(\theta, \phi) D_{ij} \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right) + D_{\Delta} \left( \begin{array}{ccc} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{array} \right) R(\theta, \phi)^T,
\]

where \(R\) is a rotation operator. The b-tensor can be parametrised in terms of \((b_j, b_l, \theta, \Phi)\):

\[
\mathbf{B} = R(\theta, \Phi) \left( \begin{array}{ccc} b_j & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right) + b_l \left( \begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{array} \right) R(\theta, \Phi)^T.
\]

Expanding \(\frac{S}{S_0}(\mathbf{B})\), using the above parameterisation, gives a separable expression in \(b_j\), and the other three parameters, \(b_l, \theta, \Phi\):

\[
\frac{S}{S_0}(b_j, b_l, \theta, \Phi) = \sum_{k=1}^{N_D} w_k \exp(-b_j (D_{ij} \Delta)) \exp(-b_l (D_{ij} \Delta)) \exp(-2b_j (D_{ij} \Delta)) \exp(-2b_l (D_{ij} \Delta) \Delta) \mathcal{P}_2 (\cos \beta_D).
\]

Taking the DTD to be expanded in a separable and orthogonal 4D basis which is the product of a 1D basis for \(b_j\) dimension and a 3D basis for \(b_l, \theta, \Phi\) dimensions. As \(\frac{S}{S_0}(\mathbf{B})\) is a function of negative exponential of \(b_j\), we use an exponential modulated with a Laguerre polynomial, which together form an orthogonal basis over \(b_j\) dimension. Similarly, we use the spherical Laguerre basis, a 3D orthogonal basis with an exponential weighting function in the radial direction, for \(b_l, \theta, \Phi\) dimensions, leading to expansion:

\[
\frac{S}{S_0}(b_j, b_l, \theta, \Phi) = \sum_{p=0}^{P} \sum_{\phi=0}^{N} \sum_{\ell=0}^{L} \sum_{n=0}^{L} \epsilon_{p\ell n \phi} Z_p(b_j) B_n^\ell(b_l, \theta, \Phi)
\]

where \(B_n^\ell(b_l, \theta, \Phi) = X_n(\theta) Y_\ell^\phi(\theta, \Phi)\) is the spherical Laguerre basis, with \(X_n(\theta) = \sqrt{\frac{2n}{\pi}} \left( \frac{\partial}{\partial \theta} \right)^n L_n^{\frac{1}{2}}(\cos \theta)\), and \(Z_p(b_j) = \frac{1}{\sqrt{2^{p}}} \exp \left( \frac{b_j \Delta}{2^{p}} \right) L_p^{\frac{1}{2}}(\frac{b_j \Delta}{2^{p}})\). \(Y_\ell^\phi(\theta, \Phi)\) are spherical harmonics of maximum degree \(L\), \(\zeta_l, \zeta_\phi\) are scale factors and \(P, N, L\) are maximum orders for \(b_j\) and \(b_l\) bases respectively. Coefficients \(\epsilon_{p\ell n \phi}\) are defined by inner product \(\epsilon_{p\ell n \phi} = \left( \frac{\partial}{\partial \theta} \right)(b_j, b_l, \theta, \Phi) \int_0^{2\pi} \int_0^\pi \frac{S}{S_0}(b_j, b_l, \theta, \Phi) Y_\ell^\phi(\theta, \Phi) \sin \theta d\theta d\phi\).
with Gauss-Laguerre quadrature\(^3\) used to choose the weights \(w_j, w_j\), and sample locations \(b_j(j), b_i(i)\) for the \(b_j\) and \(b_i\) dimensions. We use the scheme\(^{10,11}\) to place the samples in the angular dimension. For a band-limited signal, this quadrature enables exact reconstruction and efficient sampling with the number of samples equal to number of coefficients.

**Methods**

The Magic DIAMOND model is used to study maximum orders needed for expansion of the diffusion signal in the proposed basis and to evaluate reconstruction accuracy of the proposed sampling scheme. Magic DIAMOND\(^{12}\), an extension of the DIAMOND model\(^{13,14}\) for axisymmetric tensor acquisitions, describes the DTD through a mixture of non-central matrix-variate Gamma distributions, where each diffusion compartment is represented through a single distribution with two shape parameters, that set axial and radial heterogeneity of a compartment, and a scale parameter. Compartments are used to represent diffusion molecules hindered and restricted by fascicles, and free water diffusion. We used a range of diffusivities and distribution parameters representative of those in the human brain.

**Results and Discussion**

As Magic DIAMOND model is linear, it is sufficient to study each compartment individually; the truncation order of the compartment with the highest truncation order is sufficient to represent all combinations of compartments. Fig.1 shows the spherical harmonic band-limit as a function of \(b_l\). Fig.2 shows the reconstruction error as a function of increasing truncation order for maximum \(b_l\) and \(b_j\) of \(3000\mu\text{ms/mm}^2\). It was found that \(N = 3, P = 2, L(i) = [2, 4, 6, 8]\) were sufficient for reconstruction error smaller than \(SISD = 0.1\) for all compartments, resulting in the sampling scheme (Fig.3) with 192 samples, the same order as several state-of-the-art sampling schemes in q-space\(^{15}\). Fig.4 shows the mean reconstruction error evaluated at 1080 uniformly placed measurements for 640 realisations of Magic DIAMOND.

**Conclusion**

We present the first 4D basis and reconstruction algorithm for representing and reconstructing the axisymmetric b-tensor encoded diffusion signal. We study the properties of the signal to inform how many and which b-tensors should be used, thereby providing sampling recommendations. Although preliminary, the high reconstruction accuracy achieved on Magic DIAMOND model is promising for interpolation of b-tensors encoded measurements to enable algorithms for recovering the DTD to achieve higher accuracy and/or reduce the number of measurements.

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**References**


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**Figures**
Fig. 1: Maximum absolute reconstruction difference between the reconstructed signal and the ground truth as a function of the spherical harmonic band-limit $L$ and $b_1$ value. $L$ required to accurately represent (maximum absolute reconstruction difference is above $\frac{\Delta}{\Delta_0} = 0.1$, as required by MD-dMRI Gaussian diffusion assumption\(^3\)) the diffusion signal at different $b_1$-values is shown by the black line.

Fig. 2: Maximum absolute reconstruction difference between the reconstructed signal and the ground truth as a function of the $b_1$ truncation order $N$ (left) and the $b_2$ truncation order $P$ (right). The truncation order should be high enough so that the reconstruction error is smaller than $\frac{\Delta}{\Delta_0} = 0.1$ (shown by the black line).

Fig. 3: Proposed b-tensor sampling scheme: a) the b-tensors shown for a single orientation, and b) the orientations used in the proposed sampling scheme, projected onto a single sphere, samples on the inner most to outer most shell are shown in green, red (two shells with this sampling) and blue for each shell respectively. Locations where antipodal symmetry is used to infer the value of the signal are lighter in colour.
Fig.4: Mean absolute reconstruction error over 1060 uniformly placed samples on a Cartesian grid for 640 realisations of the magic DIAMOND model.